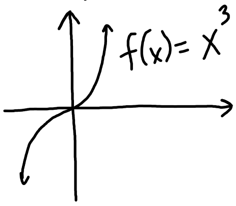
Lecture 4: Continuity

September 19, 2016   10:00 PM

A function is continuous at a number *a* if *a* is in the domain of *f* and $\lim_{x \to a} f(x) = a$.
An example of a **continuous** function:

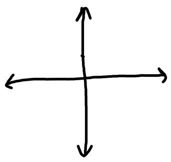


To test for continuity at a specific point:

$\lim_{x \to 0} x^3 = 0$, therefore x^3 is continuous at 0

Another, but elementary way, of viewing continuity is that the graph can be drawn "without lifting your pen", and therefore is continuous (doesn't have any breaks).

An example of a **discontinuous** function:



$$f(x) = \begin{cases} \sqrt{x-1} & x \geq 1 \\ \sqrt{-x+1} + 2 & x < 1 \end{cases}$$

$\underbrace{\lim_{x \to 1^-}}_{}$ $f(x) = 2$ BUT $f(1) = \sqrt{1-1} = 0$

"as x approaches 1 from the left"
ie. use 2nd part of definition

SO, $\underbrace{\lim_{x \to 1^-} f(x) \neq f(1)}$, f(x) isn't continuous at x=1

If *f* is defined around *a* but not at *a*, we say *f* is discontinuous at *a* (of if *f* is defined but not continuous).

<u>More examples of discontinuous functions</u>:

1) $f(x)$ from earlier
2) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$ → rational function: gap at 2 (might not be visible on graph but it *is* there
3) $f(x) = \begin{cases} \dfrac{1}{x^2} & x \neq 0 \\ 1 & x = 0 \end{cases}$
4) $f(x) = \dfrac{3x}{x^2 + 5x + 6}$ --> gaps are zeroes of denominator = -2, -3 and are also **vertical asymptotes**

A function *f* is continuous from the right at a number *a* if $\lim_{x \to a^+} f(x) = f(a)$ & from the left if $\lim_{x \to a^-} f(x) = f(a)$.

<u>Example</u>:

$$f(x) = \begin{cases} \sqrt{x-1} & x \geq 1 \\ \sqrt{-x+1} + 2, & x < 1 \end{cases}$$

$\lim_{x \to 1^-} f(x) = 2$, but:
$\lim_{x \to 1^+} f(x) = 0$, which is f(-1)

Therefore *f* is continuous from the right at 1, but not from the left

A function is continuous on an interval if it is continuous at every point in the interval.

<u>Examples of continuous functions</u>:

- polynomials

- trigonometric {sin(x), cos(x), tan(x) on (-π/2, π/2)}
- exponential functions
- root functions
- logarithmic functions

Theorem

If f and g are continuous at a and c is a constant, then the following are also continuous:

1) $f+g$ ie x^2+3e^x
2) $f-g$
3) $f \times g$ ie $x^2 e^x \sin(x)$
4) $c f$ ie $f(x)=3\sin(x)$
5) $\frac{f}{g}$ if $g(a) \neq 0$

Theorem

If f is a continuous function at b and $\lim_{x\to a} \mathrm{g}(x) = b$ then $\lim_{x\to a} \mathrm{f}(\mathrm{g}(x)) = \mathrm{f}(b)$.

In other words: $\lim_{x\to a}(f(g(x))) \underset{\uparrow}{=} f(\lim_{x\to a}(g(x))$

swap limit and a continuous function

Example:

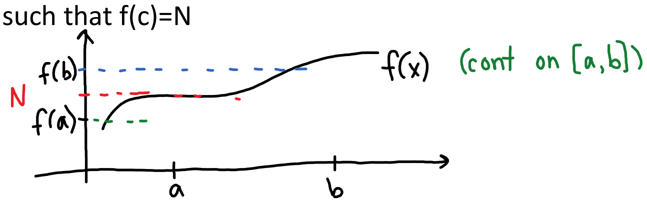
$$\lim_{x\to 1}\left(\sqrt{x^2-1}\right) = \sqrt{\lim_{x\to 1}(x^2-1)} = \sqrt{0} = 0$$

Theorem

If g is continuous at a and f is continuous at $g(a)$, then $f \circ \mathrm{g}$ is continuous at a.

The Intermediate Value Theorem

f continous on [a,b] then for any number N between f(a) and f(b) there is a point $c \in (a, b]$ such that f(c)=N



* theorem tells us that if we go right from N, we will definitely hit the graph

Limits at Infinity

$$f(x) = \frac{x^2-1}{x^2+1}$$

as x approaches infinity….

x	y
10	0.980198
100	0.999800
1000	0.999998

y gets closer and closer to 1.

Definition

If a function f is defined on (a, ∞), then $\lim_{x\to\infty} f(x) = L$ means that the values of f(x) can be made arbitrarily close to L by making x sufficiently large.

Also: $\lim_{x\to -\infty} f(x) = L$ if x is made sufficiently small.

There is a horizontal asymptote if either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$.

Examples:

1) $\lim_{x\to\infty} \frac{1}{x^2} = 0$

2) $\lim_{x\to-\infty} e^x = 0$

3) $\lim_{x\to\infty} \frac{3x^2-x-2}{5x^2+4x-1}$

$= 3/5$ ✱recall from gr 12